Some Results on Fixed Points of Asymptotically Regular Mappings

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Abstract— In this paper we extend some results of fixed points for asymptotically regular mappings on a complete 2-metric space. We generalize some fixed point theorem of 'Slobodan C. Nesic' in the context of 2-metric space
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Index Terms— Fixed Point, Asymptotically Regular Mapping, Metric space, 2-metric space, complete 2-metric space

1 Introduction

The concept of asymptotically regular at a point in a space was first introduced by Browder and Petryshyn [1]

2 Preliminaries

Definition 2.1 : A 2-metric space is a set X with a real valued non -negative function is defined on X × X × X such that

(1) for all x, y ϵ X, (x ≠ y), there exists a point z ϵ X such that σ(x, y, z) ≠ 0

(2) σ(x, y, z) = 0 if at least two of points x, y, z coincide.

(3) σ(x, y, z) = σ(x, z, y) = σ(y, z, x) = σ(y, x, z)

(4) σ(x, y, z) ≤ σ(x, y, w) + σ(x, w, z) + σ(w, y, z)

The function σ is called 2-metric for the space and (X, σ) is called a 2-metric space.

Definition 2.2 : A mapping T : X → X of a 2-metric space (X, σ) in to itself is said to be asymptotically regular at a point x ϵ X if

$$\lim_{n\to\infty} \sigma\left(T^n x, T^{n+1}x, z\right) = 0, \quad (z \in X).$$

3 Main Results

In the following theorem, we establish a unique fixed point in X

Theorem 3.1

Let (X, ϱ) be a complete 2-metric space and T : X → X be a mapping such that the following condition is satisfied.

(i) σ(Tx, Ty, z) ≤ p σ(x, y, z) + q[σ(x, Tx, z) + σ(y, Ty, z)] + r[σ(x, Ty, z) + σ(y, Tx, z)] + F[σ(x, Tx, z) · σ(y, Ty, z)]

for all x, y, z ϵ X, 0 ≤ p, r, p + 2r < 1, q + r < 1.

If T is asymptotically regular at some point of X, then T has a unique fixed point in X.

Proof: We shall assume that T is asymptotically regular at a point x_0 ϵ X and consider the sequence $\{T^n x_0\}$. Then

$$\sigma(T^m x_0, T^n x_0, z) \le p\,\sigma(T^{m-1}x_0, T^{n-1}x_0, z) + q[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] + r[\sigma(T^{m-1}x_0, T^n x_0, z) + \sigma(T^{n-1}x_0, T^m x_0, z)] + F[\sigma(T^{m-1}x_0, T^m x_0, z)\cdot\sigma(T^{n-1}x_0, T^n x_0, z)]$$

$$\le p\,[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^m x_0, T^{n-1}x_0, z) + \sigma(T^{m-1}x_0, T^{n-1}x_0, T^m x_0)] + q[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] + r[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^m x_0, T^n x_0, z) + \sigma(T^{m-1}x_0, T^n x_0, T^m x_0) + \sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(T^n x_0, T^m x_0, z) + \sigma(T^{n-1}x_0, T^m x_0, T^n x_0)] + F[\sigma(T^{m-1}x_0, T^m x_0, z)\cdot\sigma(T^{n-1}x_0, T^n x_0, z)]$$

$$\le p[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^m x_0, T^n x_0, z) + \sigma(T^n x_0, T^{n-1}x_0, z) + \sigma(T^m x_0, T^{n-1}x_0, T^n x_0) + \sigma(T^{m-1}x_0, T^{n-1}x_0, T^m x_0)] + q[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] + 2r\,\sigma(T^m x_0, T^n x_0, z) + r[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] + F[\sigma(T^{m-1}x_0, T^m x_0, z)\cdot\sigma(T^{n-1}x_0, T^n x_0, z)]$$

i.e. $\sigma(T^m x_0, T^n x_0, z)$

$$\le \frac{(p+q+r)}{1-p-2r}[\sigma(T^{m-1}x_0, T^m x_0, z) + \sigma(T^{n-1}x_0, T^n x_0, z)] + \frac{1}{(1-p-2r)}F[\sigma(T^{m-1}x_0, T^m x_0, z)\cdot\sigma(T^{n-1}x_0, T^n x_0, z)]$$

Since T is asymptotically regular at x_0,

$\sigma(T^m x_0, T^n x_0, z) \to 0$ as m , n → ∞, (z ϵ X)

∴ $\sigma(T^{m-1}x_0, T^{n-1}x_0, T^m x_0) \to 0$

$\sigma(T^{m-1}x_0, T^n x_0, T^m x_0) \to 0$

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$\sigma(T^{n-1}x_0, T^m x_0, T^n x_0) \to 0$ etc as m, n → ∞

Hence $\{T^n x_0\}$ is a Cauchy sequence,

Since (X, σ) is complete, there exist a point u ϵ X such that $u = \lim_{n \to \infty} T^n x_0$

Suppose that u is not a fixed point of T.

Then by (i), we obtain

$\sigma(u,Tu,z) \leq \sigma(u,T^n x_0, z) + \sigma(T^n x_0,Tu, z)$

$\leq \sigma(u, T^n x_0, z) + p[\sigma(T^{n-1}x_0, u, z)] + q[\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(u,Tu, z)] + r[\sigma(T^{n-1}x_0, Tu, z) + \sigma(u, T^n x_0, z)] + F[\sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(u, Tu, z)]$

$\leq \sigma(u, T^n x_0, z) + p[\sigma(T^{n-1}x_0, u, z)] + q[\sigma(T^{n-1}x_0, T^n x_0, z) + \sigma(u, Tu, z)] + r\,\sigma(u,T^n x_0, z) + r[\sigma(u, Tu, z) + \sigma(T^{n-1}x_0, u, z) + \sigma(T^{n-1}x_0, Tu, z)] + F[\sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(u, Tu, z)]$

$= (1 + r)\,\sigma(u, T^n x_0, z) + (p + r)\,\sigma(u, T^{n-1}x_0, z)$ ` $+ q\,\sigma(T^{n-1}x_0, T^n x_0, z) + (q + r)\,\sigma(u, Tu, z) + F[\sigma(T^{n-1}x_0, T^n x_0, z) \cdot \sigma(u, Tu, z)]$

Taking the limit as n → ∞, we obtain $\sigma(u, Tu, z) \leq (q + r)\,\sigma(u, Tu, z)$

Which contradicts $(q + r) < 1$ unless u = Tu,

Suppose T has second fixed point v in X. Then by (i),

We obtain $\sigma(u,v, z) \leq (p +2r)\,\sigma(u, v, z)$

Since $(p + 2r) < 1$ it follows that u = v.

Hence the fixed point is unique.

Theorem 3.2: Let (X, σ) be a 2-metric space and T be a mapping of X into itself satisfying the condition

(i) $\sigma(Tx,Ty, z) \leq p\sigma(x, y, z) + q[\sigma(x,Tx, z) + \sigma(y,Ty, z)] + r[\sigma(x,Ty, z) + \sigma(y,Tx, z)] + F[\sigma(x,Tx, z) \cdot \sigma(y,Ty, z)]$ for all x, y, z ϵ X, $0 \leq p, r, p + 2r < 1, q + r < 1$.

If T is asymptotically regular at a point x in X and sequence of iterates $\{T^n x\}$ has a subsequence converging to a point z ϵ X, then z is a unique fixed point of T and $\{T^n x\}$ are also converges to z.

Proof:

Let T be asymptotically regular at x ϵ X and consider the sequence $\{T^n x\}$, we shall assume that $\lim_{k \to \infty} T_{n_k} = z$ and Tz ≠ z.

Then by condition (i), we obtain, (for u ϵ X)

$\sigma(z, Tz, u)$

$\leq \sigma(z, T^{n_k}x, u) + \sigma(T^{n_k}x, T^{n_k+1}x, u) + \sigma(T^{n_k+1}x, Tz, u)$

$\leq \sigma(z, T^{n_k}x, u) + \sigma(T^{n_k}x, T^{n_k+1}x, u) + p\,\sigma(T^{n_k}x, z, u) + q[\sigma(T^{n_k}x, T^{n_k+1}x, u) + \sigma(z, Tz, u)] + r[\sigma(z, T^{n_k+1}x, u) + \sigma(T^{n_k}x, Tz, u)] + F[\sigma(T^{n_k}x, T^{n_k+1}x, u) \cdot \sigma(z, Tz, u)]$

$\leq \sigma(z, T^{n_k}x, u) + \sigma(T^{n_k}x, T^{n_k+1}x, u) + p\,\sigma(T^{n_k}x, z, u) + q[\sigma(T^{n_k}x, T^{n_k+1}x, u) + \sigma(z, Tz, u)] + r[\sigma(z, T^{n_k+1}x, u) + \sigma(z, Tz, u) + \sigma(T^{n_k}x, z, u) + \sigma(T^{n_k}x, Tz, z)] + F[\sigma(T^{n_k}x, T^{n_k+1}x, u) \cdot \sigma(z, Tz, u)]$

$\leq \sigma(z, T^{n_k}x, u) + \sigma(T^{n_k}x, T^{n_k+1}x, u) + p\,\sigma(T^{n_k}x, z, u) + q[\sigma(T^{n_k}x, T^{n_k+1}x, u) + \sigma(z, Tz, u)] + r[\sigma(z, T^{n_k+1}x, u) + \sigma(z, Tz, u) + \sigma(T^{n_k}x, Tz, z) + \sigma(T^{n_k}x, z, u)] + F[\sigma(T^{n_k}x, T^{n_k+1}x, u) \cdot \sigma(z, Tz, u)]$

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Taking limit as $k \to \infty$, we obtain

$\sigma(z, Tz, u) \leq (q + r)\, \sigma(z, Tz, u)$ which is contradicts $q + r < 1$ unless $z = Tz$.
By Theorem 3.1, z is the unique fixed point. By using (i), we obtain

$$\sigma(z, T^n x, u) = \sigma(Tz, T^n z, u)$$
$$\leq \sigma(Tz, T^{n+1}x, u) + \sigma(T^{n+1}x, T^n x, u) + \sigma(Tz, T^n x, T^{n+1}x).$$

$$\therefore \sigma(z, T^n x, u)$$
$$\leq \sigma(Tz, T^{n+1}x, u) + \sigma(T^{n+1}x, T^n x, u)$$

$$\leq p\, \sigma(z, T^n x, u) + q\, \sigma(z, Tz, u) + q\, \sigma(T^n x, T^{n+1}x, u) + r[\sigma(z, T^{n+1}x, u) + \sigma(T^n x, Tz, u)\} + F[\sigma(z, Tz, u) \cdot \sigma(T^n x, T^{n+1}x, u)]$$

$$\leq p\, \sigma(z, T^n x, u) + q[\sigma(z, Tz, u) + \sigma(T^n x, T^{n+1}x, u)] + r[\sigma(z, T^n x, u) + \sigma(T^n x, T^{n+1}x, u) + \sigma(z, T^{n+1}x, T^n x)] + \sigma(z, Tz, u) + \sigma(T^n x, z, u) + \sigma(T^n x, Tz, z) + \sigma(T^{n+1}x, T^n x, u)$$

$$\leq p\, \sigma(z, T^n x, u) + q[\sigma(z, Tz, u) + \sigma(T^n x, T^{n+1}x, u)] + r[\sigma(z, T^n x, u) + \sigma(T^n x, T^{n+1}x, u) + \sigma(z, Tz, u) + \sigma(T^n x, z, u)] + F[\sigma(z, Tz, u) \cdot \sigma(T^n x, T^{n+1}x, u)] + \sigma(T^{n+1}x, T^n x, u)$$

This implies that

$$\sigma(z, T^n x, u) \leq \frac{(1+q+r)}{(1-p-2r)}\, \sigma(T^n x, T^{n+1}x, u), \qquad (x \in X)$$

since $p + 2r < 1$, $Tz = z$,
since T is asymptotically regular, $\lim_{n\to\infty} \sigma(z, T^n x, u) = 0$.
This implies that $\{T^n x\}$ converges to z.

This completes the proof.

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